CONVERGENCE OF THE FULL COMPRESSIBLE NAVIER-STOKES-MAXWELL SYSTEM TO THE INCOMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS IN A BOUNDED DOMAIN

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Abstract. In this paper we establish the uniform estimates of strong solutions with respect to the Mach number and the dielectric constant to the full compressible Navier-Stokes-Maxwell system in a bounded domain. Based on these uniform estimates, we obtain the convergence of the full compressible Navier-Stokes-Maxwell system to the incompressible magnetohydrodynamic equations for well-prepared data.

1. Introduction

In this paper we consider the singular limit of the following full compressible Navier-Stokes-Maxwell system in a bounded domain $\Omega \subset \mathbb{R}^3$ ([5]):

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \frac{1}{\epsilon_1^2}\nabla p - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u \\ = (E + u \times b) \times b,\end{aligned} \tag{1.2}$$

$$\begin{aligned}\partial_t(\rho e) + \operatorname{div}(\rho u e) + p \operatorname{div} u - \operatorname{div}(\kappa \nabla \mathcal{T}) \\ = \epsilon_1^2(2\mu|D(u)|^2 + \lambda(\operatorname{div} u)^2 + (E + u \times b)^2),\end{aligned} \tag{1.3}$$

$$\epsilon_2 \partial_t E - \operatorname{rot} b + E + u \times b = 0, \tag{1.4}$$

$$\partial_t b + \operatorname{rot} E = 0, \ \operatorname{div} b = 0, \tag{1.5}$$

where the unknowns $\rho, u, p, e, \mathcal{T}, E$, and b stand for the density, velocity, pressure, internal energy, temperature, electric field, and magnetic field, respectively. The

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physical constants μ and λ are the shear viscosity and bulk viscosity of the flow and satisfy $\mu > 0$ and $\lambda + \frac{2}{3}\mu \geq 0$. $\kappa > 0$ is the heat conductivity. $\epsilon_1 > 0$ is the (scaled) Mach number, and $\epsilon_2 > 0$ is the (scaled) dielectric constant. $D(u) := \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$, where ∇u^{T} denotes the transpose of the matrix ∇u.

In [8, 9], Kawashima and Shizuta established the global existence of smooth solutions for small data [11] and studied its zero dielectric constant limit $\epsilon_2 \to 0$ in the whole space $\mathbb{R}^2$. Recently, Jiang and Li [6] studied the zero dielectric constant limit $\epsilon_2 \to 0$ to the system (1.1)-(1.5) and obtained the convergence of the system (1.1)-(1.5) to the full compressible magnetohydrodynamic equations in $\mathbb{T}^3$, see also [7] on the similar results to the invisid case of (1.1)-(1.5). In [10], Li and Mu study the low Mach number limit $\epsilon_1 \to 0$ to the system (1.1)-(1.5) and obtained the convergence of the system (1.1)-(1.5) to the incompressible Navier-Stokes-Maxwell system in the torus $\mathbb{T}^3$.

It should be pointed out that no boundary effect is considered in the references mentioned above. The purpose of this paper is to invistigate the singular limit $\epsilon_1, \epsilon_2 \to 0$ to the system (1.5)-(1.5) in a bounded domain. For simplicity, we shall take $\epsilon_1 = \epsilon_2 = \epsilon$ and consider the case that the fluid is a polytropic ideal gas, that is

$$e := C_V \mathcal{T}, \quad p := \mathcal{R}\rho\mathcal{T} \tag{1.6}$$

with $C_V > 0$ and $\mathcal{R}$ being the specific heat at constant volume and the generic gas constant, respectively.

To state the main result of this paper, we denote the density and temperature variations by σ^ϵ and θ^ϵ:

$$\rho^\epsilon := 1 + \epsilon\sigma^\epsilon, \ \mathcal{T}^\epsilon := 1 + \epsilon\theta^\epsilon. \tag{1.7}$$

Then we can rewrite the system (1.1)-(1.5) as follows:

$$\partial_t \sigma^\epsilon + \operatorname{div}(\sigma^\epsilon u^\epsilon) + \frac{1}{\epsilon}\operatorname{div} u^\epsilon = 0, \tag{1.8}$$

$$\begin{aligned} &\rho^\epsilon(\partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon) + \frac{\mathcal{R}}{\epsilon}(\nabla\sigma^\epsilon + \nabla\theta^\epsilon) + \mathcal{R}\nabla(\sigma^\epsilon\theta^\epsilon) - \mu\Delta u^\epsilon - (\lambda+\mu)\nabla \operatorname{div} u^\epsilon \\ &\qquad = (E^\epsilon + u^\epsilon \times b^\epsilon) \times b^\epsilon, \end{aligned} \tag{1.9}$$

$$\begin{aligned} &C_V \rho^\epsilon(\partial_t\theta^\epsilon + u^\epsilon \cdot \nabla\theta^\epsilon) + \mathcal{R}(\rho^\epsilon\theta^\epsilon + \sigma^\epsilon)\operatorname{div} u^\epsilon + \frac{\mathcal{R}}{\epsilon}\operatorname{div} u^\epsilon \\ &\qquad = \kappa\Delta\theta^\epsilon + \epsilon[2\mu|D(u^\epsilon)|^2 + \lambda(\operatorname{div} u^\epsilon)^2 + (E^\epsilon + u^\epsilon \times b^\epsilon)^2], \end{aligned} \tag{1.10}$$

$$\epsilon\partial_t E^\epsilon - \operatorname{rot} b^\epsilon + E^\epsilon + u^\epsilon \times b^\epsilon = 0, \tag{1.11}$$

$$\partial_t b^\epsilon + \operatorname{rot} E^\epsilon = 0, \ \operatorname{div} b^\epsilon = 0. \tag{1.12}$$

Here we have added the superscript ϵ on the unknowns $(\sigma, u, \theta, E, b)$ to emphasise the dependence of ϵ. The system (1.8)-(1.12) are supplemented with the following initial and boundary conditions:

$$(\sigma^\epsilon, u^\epsilon, \theta^\epsilon, E^\epsilon, b^\epsilon)(\cdot, 0) = (\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon)(\cdot) \ \text{ in } \ \Omega, \tag{1.13}$$

$$u^\epsilon \cdot n = 0, \ \operatorname{rot} u^\epsilon \times n = 0, \ \frac{\partial \theta^\epsilon}{\partial n} = 0, \ E^\epsilon \times n = 0, \ b^\epsilon \cdot n = 0 \ \text{ on } \ \partial\Omega, \tag{1.14}$$

where n is the unit outer normal vector to the smooth boundary $\partial\Omega$.

Formally, if we let $\epsilon \to 0$ in (1.8) and (1.9), then we obtain that $\operatorname{div} u^\epsilon \to 0, \nabla\theta^\epsilon \to 0$, and $\nabla\sigma^\epsilon \to 0$. Letting $\epsilon = 0$ in (1.11) gives $E^\epsilon = \operatorname{rot} b^\epsilon - u^\epsilon \times b^\epsilon$. Pulling it into (1.12) and taking the limit $\epsilon \to 0$ we obtain the following limit system (suppose that the limits $(u^\epsilon, b^\epsilon) \to (v, B)$ exist):

$$\begin{cases} v_t + v \cdot \nabla v + \nabla \pi - \mu \Delta v = \operatorname{rot} B \times B, \\ B_t + \operatorname{rot}(B \times v) - \Delta B = 0, \\ \operatorname{div} v = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{1.15}$$

We shall give a rigorous proof the the above formal analysis below.

Denote

$$\begin{aligned} M^\epsilon(t) := \sup_{0 \le s \le t} \Big\{ & \|(\sigma^\epsilon, u^\epsilon, \theta^\epsilon, \sqrt{\epsilon} E^\epsilon, b^\epsilon)(\cdot, s)\|_{H^2} + \|\partial_t(\sigma^\epsilon, u^\epsilon, \theta^\epsilon, \sqrt{\epsilon} E^\epsilon, b^\epsilon)(\cdot, s)\|_{H^1} \\ & + \epsilon \|\partial_t^2 (\sigma^\epsilon, u^\epsilon, \theta^\epsilon)(\cdot, s)\|_{L^2} + \Big\| \frac{1}{1 + \epsilon \sigma^\epsilon(\cdot, s)} \Big\|_{L^\infty} \Big\} \\ & + \Big\{ \int_0^t \Big(\|(u^\epsilon, \theta^\epsilon)\|_{H^3}^2 + \|\partial_t (u^\epsilon, \theta^\epsilon)\|_{H^2}^2 + \|\epsilon \partial_t^2 (\sigma^\epsilon, u^\epsilon, \theta^\epsilon)\|_{H^1}^2 \\ & + \|E^\epsilon\|_{H^2}^2 + \|\partial_t (E^\epsilon, b^\epsilon)\|_{H^1}^2 \Big) ds \Big\}^{\frac{1}{2}}. \end{aligned} \tag{1.16}$$

First, we have

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a simply connected, bounded domain with smooth boundary $\partial\Omega$ and $0 < \epsilon < 1$. Suppose that the initial data $(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon)$ satisfy the following regularity conditions:*

$$0 \le \theta_0^\epsilon, \quad 0 < \frac{1}{K_0} \le 1 + \epsilon \sigma_0^\epsilon \le K_0, \tag{1.17}$$

$$\begin{aligned} & \|(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon)\|_{H^2} + \|\partial_t(\sigma^\epsilon, u^\epsilon, \theta^\epsilon, E^\epsilon, b^\epsilon)(\cdot, 0)\|_{H^1} \\ & \qquad + \epsilon \|\partial_t^2 (\sigma^\epsilon, u^\epsilon, \theta^\epsilon)(\cdot, 0)\|_{L^2} \le K_1 \end{aligned} \tag{1.18}$$

for some positive constants $K_0 > 1$ and K_1 independent of $\epsilon > 0$. Then there exist a small time $\tilde{T} > 0$ independent of $\epsilon > 0$ and a unique strong solution $(\sigma, u, \theta, E, b)$ to the initial boundary value problem (1.8)-(1.14) *such that*

$$M^\epsilon(\tilde{T}) \leq K \tag{1.19}$$

for some positive constant K independent of $\epsilon > 0$.

Remark 1.1. *In the assumption* (1.18), *$\sigma_t^\epsilon(\cdot, 0)$ is indeed defined by $-\mathrm{div}\,(\sigma_0^\epsilon u_0^\epsilon) + \frac{1}{\epsilon}\mathrm{div}\, u_0^\epsilon$ through the density equation and the other quantities are defined by an analogous way.*

Based on the uniform estimates of the solutions, we can prove the following convergence result by applying the Arzelá-Ascolis theorem in a standard way.

Theorem 1.2. *Let $(\sigma^\epsilon, u^\epsilon, \theta^\epsilon, E^\epsilon, b^\epsilon)$ be the solution of the problem* (1.8)-(1.14) *with initial data $(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon)$ satisfying the conditions in Theorem 1.1. Assume further that the initial data $(\sigma_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon, E_0^\epsilon, b_0^\epsilon)$ satisfy that*

$$(\epsilon\sigma_0^\epsilon, u_0^\epsilon, \epsilon\theta_0^\epsilon, b_0^\epsilon) \to (0, v_0, 0, B_0) \text{ strongly in } H^s \text{ for any } 0 \leq s < 2 \quad \text{as} \quad \epsilon \to 0,$$

$$E_0^\epsilon \to \mathrm{rot}\, B_0 - v_0 \times B_0 \text{ strongly in } H^s \text{ for any } 0 \leq s < 1 \quad \text{as} \quad \epsilon \to 0.$$

Then $(\epsilon\sigma^\epsilon, u^\epsilon, \epsilon\theta^\epsilon, b^\epsilon) \to (0, v, 0, B)$ strongly in $L^\infty(0, \tilde{T}; H^1)$ and $E^\epsilon \to B - v \times B$ strongly in $L^\infty(0, \tilde{T}; L^2)$ as $\epsilon \to 0$, where (v, B) satisfies (1.15) *with the following initial and boundary conditions:*

$$\begin{cases} v \cdot n = B \cdot n = 0, \ \mathrm{rot}\, v \times n = \mathrm{rot}\, B \times n = 0 \quad on \quad \partial\Omega \times (0, \tilde{T}], \\ (v, B)(\cdot, 0) = (v_0, B_0)(\cdot) \quad in \quad \Omega \subseteq \mathbb{R}^3. \end{cases} \tag{1.20}$$

The remainder of this paper is devoted to the proof of Theorem 1.1 which will be given in next section.

2. Proof of Theorem 1.1

In this section we shall prove Theorem 1.1 by combining the ideas developed in [1, 3, 4, 11]. First, by taking the similar arguments to that [1, 11], we know that in order to prove (1.19), it suffices to show the following inequality

$$M^\epsilon(t) \leq C_0(M^\epsilon(0)) \exp[t^{\frac{1}{4}} C(M^\epsilon(t))] \tag{2.1}$$

for $\forall t \in [0, \tilde{T}]$ and some given positive nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

Below we shall omit the spatial domain Ω in the integrals and drop the superscript "ϵ" of $\rho^\epsilon, \sigma^\epsilon, u^\epsilon, \theta^\epsilon$, etc. for the sake of simplicity; moreover, we write $M^\epsilon(t)$ and $M^\epsilon(0)$ as M and M_0, respectively. Since the physical constants κ, C_V, and $\mathcal{R}$ do not bring any essential difficulties in our arguments, we shall take $\kappa = C_V = \mathcal{R} = 1$.

We will also use the following two inequalities:

$$\|u\|_{H^s(\Omega)} \le C(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\operatorname{rot} u\|_{H^{s-1}(\Omega)} + \|u\|_{H^{s-1}(\Omega)} + \|u \cdot n\|_{H^{s-1/2}(\partial\Omega)}), \tag{2.2}$$

$$\|u\|_{H^s(\Omega)} \le C(\|\operatorname{div} u\|_{H^{s-1}(\Omega)} + \|\operatorname{rot} u\|_{H^{s-1}(\Omega)} + \|u\|_{H^{s-1}(\Omega)} + \|u \times n\|_{H^{s-1/2}(\partial\Omega)}), \tag{2.3}$$

for any $u \in H^s(\Omega)$ with $s \ge 1$, which were obtained in [2] and [12] respectively.

Because the local existence for the problem (1.8)-(1.14) with fixed $\epsilon > 0$ is essential similar to that in [13], we only need to prove (2.1). We will use the methods developed in [3, 4].

First, by the same calculations as that in [3], we get

$$\Big\|\frac{1}{\rho}(\cdot,t)\Big\|_{L^\infty} + \|\rho(\cdot,t)\|_{H^2} \le C_0(M_0)\exp(C\sqrt{t}M), \tag{2.4}$$

$$\|\rho_t(\cdot,t)\|_{H^1} \le C(M). \tag{2.5}$$

Now we use the same method as that in [4] to prove some a priori estimates on (E, b).

Testing (1.11) and (1.12) by E and b, respectively, and summing up the results, we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int(\epsilon E^2 + b^2)dx + \int E^2 dx &= \int (b \times u) E dx \\ &\le \|b\|_{L^2}\|E\|_{L^2}\|u\|_{L^\infty} \le \frac{1}{2}\int E^2 dx + C(M).\end{aligned}$$

Integrating the above inequality over $(0, t)$, we find that

$$\int(\epsilon E^2 + b^2)dx + \int_0^t \int E^2 dxds \le C_0(M_0) + tC(M). \tag{2.6}$$

Using (1.14) and the formula

$$-(u \times b) \times n = (b \cdot n)u - (n \cdot u)b = 0 \quad \text{on} \quad \partial\Omega, \tag{2.7}$$

we infer that

$$\operatorname{rot} b \times n = 0 \quad \text{on} \quad \partial\Omega. \tag{2.8}$$

Taking rot to (1.11) and (1.12), testing the results by $\mathrm{rot}\, E$ and $\mathrm{rot}\, b$, respectively, summing up the results, and using (2.8) and integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\epsilon|\mathrm{rot}\, E|^2+|\mathrm{rot}\, b|^2)dx+\int|\mathrm{rot}\, E|^2dx\\
&\quad=-\int \mathrm{rot}\,(u\times b)\cdot\mathrm{rot}\, E dx\\
&\quad\le\frac{1}{2}\int|\mathrm{rot}\, E|^2dx+C\|u\|_{H^2}^2\|\mathrm{rot}\, b\|_{L^2}^2\\
&\quad\le\frac{1}{2}\int|\mathrm{rot}\, E|^2dx+C(M).
\end{aligned}$$

Integrating the above inequality over $(0,t)$, we have

$$\int(\epsilon|\mathrm{rot}\, E|^2+|\mathrm{rot}\, b|^2)dx+\int_0^t\int|\mathrm{rot}\, E|^2dxds\le C_0(M_0)+tC(M). \tag{2.9}$$

Taking div to (1.11) and testing the result by $\mathrm{div}\, E$, we infer that

$$\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int(\mathrm{div}\, E)^2dx+\int(\mathrm{div}\, E)^2dx=\int\mathrm{div}\,(b\times u)\mathrm{div}\, E dx\\
&\quad\le\frac{1}{2}\int(\mathrm{div}\, E)^2dx+C(M).
\end{aligned}$$

Integrating the above inequality over (0,t), we deduce that

$$\epsilon\int(\mathrm{div}\, E)^2dx+\int_0^t\int(\mathrm{div}\, E)^2dxds\le C_0(M_0)+tC(M). \tag{2.10}$$

Taking ∂_t to (1.11) and (1.12), testing the results by E_t and b_t, respectively, summing up the results, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int(\epsilon|E_t|^2+|b_t|^2)dx+\int|E_t|^2dx&=\int\partial_t(b\times u)\partial_t E dx\\
&\le(\|b_t\|_{L^2}\|u\|_{L^\infty}+\|b\|_{L^\infty}\|u_t\|_{L^2})\|E_t\|_{L^2}\\
&\le\frac{1}{2}\int|E_t|^2dx+C(M).
\end{aligned}$$

Integrating the above inequality over $(0,t)$, we get

$$\int(\epsilon|E_t|^2+|b_t|^2)dx+\int_0^t\int|E_t|^2dxds\le C_0(M_0)+tC(M). \tag{2.11}$$

(1.12) and (2.8) give the boundary condition

$$\mathrm{rot}^{\,2}E\times n=0\ \text{ on }\ \partial\Omega. \tag{2.12}$$

Taking $\mathrm{rot}^{\,2}$ to (1.11) and (1.12), testing the results by $\mathrm{rot}^{\,2}E$ and $\mathrm{rot}^{\,2}b$, respectively, summing up the results, we derive that

$$\frac{1}{2}\frac{d}{dt}\int(\epsilon|\mathrm{rot}^{\,2}E|^2+|\mathrm{rot}^{\,2}b|^2)dx+\int|\mathrm{rot}^{\,2}E|^2dx$$

$$
\begin{aligned}
&= \int \mathrm{rot}^2(b\times u)\mathrm{rot}^2 E dx\\
&\le \frac{1}{2}\int |\mathrm{rot}^2 E|^2 dx + C\|b\|_{H^2}^2\|u\|_{H^2}^2\\
&\le \frac{1}{2}\int |\mathrm{rot}^2 E|^2 dx + C(M).
\end{aligned}
$$

Integrating the above inequality over $(0,t)$, we have

$$
\int (\epsilon|\mathrm{rot}^2 E|^2 + |\mathrm{rot}^2 b|^2)dx + \int_0^t\int |\mathrm{rot}^2 E|^2 dxds \le C_0(M_0) + tC(M). \tag{2.13}
$$

Taking $\nabla\mathrm{div}$ to (1.11), testing the result by $\nabla\mathrm{div}\, E$, we have

$$
\begin{aligned}
\frac{\epsilon}{2}\frac{d}{dt}\int |\nabla\mathrm{div}\, E|^2 dx + \int |\nabla\mathrm{div}\, E|^2 dx &= \int \nabla\mathrm{div}\,(b\times u)\cdot\nabla\mathrm{div}\, E dx\\
&\le \frac{1}{2}\int |\nabla\mathrm{div}\, E|^2 dx + C\|b\|_{H^2}^2\|u\|_{H^2}^2\\
&\le \frac{1}{2}\int |\nabla\mathrm{div}\, E|^2 dx + C(M).
\end{aligned}
$$

Integrating the above inequality over $(0,t)$, we obtain

$$
\epsilon\int |\nabla\mathrm{div}\, E|^2 dx + \int_0^t\int |\nabla\mathrm{div}\, E|^2 dxds \le C_0(M_0) + tC(M). \tag{2.14}
$$

Taking $\partial_t\mathrm{rot}$ to (1.11) and (1.12), testing the results by $\partial_t\mathrm{rot}\, E$ and $\partial_t\mathrm{rot}\, b$, respectively, summing up the results, and using (2.8), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (\epsilon|\mathrm{rot}\, E_t|^2 + |\mathrm{rot}\, b_t|^2)dx + \int |\mathrm{rot}\, E_t|^2 dx\\
&= \int \mathrm{rot}\,(b_t\times u + b\times u_t)\mathrm{rot}\, E_t dx\\
&\le C(\|b_t\|_{H^1}\|u\|_{H^2} + \|b\|_{H^2}\|u_t\|_{H^1})\|\mathrm{rot}\, E_t\|_{L^2}\\
&\le \frac{1}{2}\int |\mathrm{rot}\, E_t|^2 dx + C(M).
\end{aligned}
$$

Integrating the above inequality over $(0,t)$, we obtain

$$
\int (\epsilon|\mathrm{rot}\, E_t|^2 + |\mathrm{rot}\, b_t|^2)dx + \int_0^t\int |\mathrm{rot}\, E_t|^2 dxds \le C_0(M_0) + tC(M). \tag{2.15}
$$

Applying $\partial_t\mathrm{div}$ to (1.11), testing the result by $\mathrm{div}\, E_t$, we have

$$
\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int (\mathrm{div}\, E_t)^2 dx + \int (\mathrm{div}\, E_t)^2 dx\\
&= \int \mathrm{div}\,(b_t\times u + b\times u_t)\mathrm{div}\, E_t dx\\
&\le C(\|b_t\|_{H^1}\|u\|_{H^2} + \|u_t\|_{H^1}\|b\|_{H^2})\|\mathrm{div}\, E_t\|_{L^2}\\
&\le \frac{1}{2}\int (\mathrm{div}\, E_t)^2 dx + C(M).
\end{aligned}
$$

Integrating the above inequality over $(0,t)$, we have

$$\epsilon\int(\operatorname{div}E_t)^2dx+\int_0^t\int(\operatorname{div}E_t)^2dxds\leq C_0(M_0)+tC(M). \tag{2.16}$$

Now we use the method in [3] to prove some a priori estimates on (σ,u,θ). Testing (1.8), (1.9) and (1.10) by σ,u and θ, respectively, summing up the results, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\sigma^2+\rho u^2+\rho\theta^2)dx+\int(\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div}u)^2+|\nabla\theta|^2)dx\\
=&\int\operatorname{div}u\left(-\frac{1}{2}\sigma^2-\rho\theta^2\right)dx+\epsilon\int\theta(2\mu|D(u)|^2+\lambda(\operatorname{div}u)^2+(E+u\times b)^2)dx\\
\leq&\|\nabla u\|_{L^\infty}(\|\sigma\|_{L^2}^2+\|\rho\|_{L^\infty}\|\theta\|_{L^2}^2)\\
\leq&+C\|\nabla u\|_{L^\infty}(\|\nabla u\|_{L^2}+\|E\|_{L^4}^2+\|u\|_{L^\infty}^2\|b\|_{L^4}^2)\|\theta\|_{L^2}\\
\leq&\|\nabla u\|_{L^\infty}C(M)\leq\|u\|_{H^3}C(M).
\end{aligned}$$

Integrating the above inequality over $(0,t)$, we obtain

$$\int(\sigma^2+\rho u^2+\rho\theta^2)dx+\int_0^t\int(|\nabla u|^2+|\nabla\theta|^2)dxds\leq C_0(M_0)+\sqrt{t}C(M). \tag{2.17}$$

Applying ∂_t to (1.8), (1.9) and (1.10), we see that

$$\partial_{tt}+\frac{1}{\epsilon}\operatorname{div}u_t=-\operatorname{div}(\sigma u)_t, \tag{2.18}$$

$$\begin{aligned}
&\rho(u_{tt}+u\cdot\nabla u_t)+\frac{1}{\epsilon}(\nabla\sigma_t+\nabla\theta_t)-\mu\Delta u_t-(\lambda+\mu)\nabla\operatorname{div}u_t\\
&\qquad=-\rho_tu_t-(\rho u)_t\nabla u-\nabla(\sigma\theta)_t+[(E+u\times b)\times b]_t,
\end{aligned}\tag{2.19}$$

$$\begin{aligned}
&\rho(\theta_{tt}+u\cdot\nabla\theta_t)+\frac{1}{\epsilon}\operatorname{div}u_t-\Delta\theta_t=-\rho_t\theta_t-(\rho u)_t\cdot\nabla\theta-((\rho\theta+\sigma)\operatorname{div}u)_t\\
&\qquad+\epsilon(2\mu|D(u)|^2+\lambda(\operatorname{div}u)^2+(E+u\times b)^2)_t.
\end{aligned}\tag{2.20}$$

Testing (2.18), (2.19) and (2.20) by σ_t,u_t and θ_t, respectively, summing up the results, we reach

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\sigma_t^2+\rho u_t^2+\rho\theta_t^2)dx+\int(\mu|\nabla u_t|^2+(\lambda+\mu)(\operatorname{div}u_t)^2+|\nabla\theta_t|^2)dx\\
&\quad=\int(\sigma u)_t\nabla\sigma_tdx-\int[\rho_tu_t+(\rho u)_t\nabla u+\nabla(\sigma\theta)_t]u_tdx\\
&\quad+\int((E+u\times b)\times b)_tu_tdx-\int[\rho_t\theta_t+(\rho u)_t\nabla\theta+((\rho\theta+\sigma)\operatorname{div}u)_t]\theta_tdx\\
&\quad+\epsilon\int(2\mu|D(u)|^2+\lambda(\operatorname{div}u)^2+(E+u\times b)^2)_t\theta_tdx\\
&\quad\leq C(M)+\int((E+u\times b)\times b)_tu_tdx+\epsilon\int((E+u\times b)^2)_t\theta_tdx
\end{aligned}$$

$$\leq C(M) + \|E_t\|_{L^2} C(M).$$

Integrating the above inequality over $(0,t)$, we have

$$\int (\sigma_t^2 + \rho u_t^2 + \rho \theta_t^2) dx + \int_0^t \int (|\nabla u_t|^2 + |\nabla \theta_t|^2) dx ds \leq C_0(M_0) + \sqrt{t} C(M). \quad (2.21)$$

Testing (2.19) by $-\nabla \mathrm{div}\, u$ in $L^2(\Omega \times (0,t))$, we find that

$$\begin{aligned}
&\frac{\mu+\lambda}{2} \|\nabla \mathrm{div}\, u(\cdot,t)\|_{L^2}^2 - \frac{1}{\epsilon} \int_0^t \int (\nabla \sigma_t + \nabla \theta_t) \cdot \nabla \mathrm{div}\, u dx ds \\
=&\frac{\mu+\lambda}{2} \|\nabla \mathrm{div}\, u_0\|_{L^2}^2 + \int_0^t \int \rho (u_{tt} + u \cdot \nabla u_t) \nabla \mathrm{div}\, u dx ds \\
&+ \int_0^t \int (\rho_t u_t + (\rho u)_t \nabla u + \nabla(\sigma\theta)_t) \cdot \nabla \mathrm{div}\, u dx ds \\
&- \int_0^t \int ((E + u \times b) \times b)_t \nabla \mathrm{div}\, u dx ds \\
=:& \frac{\mu+\lambda}{2} \|\nabla \mathrm{div}\, u_0\|_{L^2}^2 + I_1 + I_2 + I_3. \quad (2.22)
\end{aligned}$$

We bound I_1, I_2 and I_3 as follows.

$$\begin{aligned}
I_1 \leq& C(M) \int_0^t \|u_{tt}\|_{L^2} ds + t C(M) \leq \sqrt{t} C(M), \\
I_2 \leq& t C(M), \\
I_3 \leq& C(M) \int_0^t \|E_t\|_{H^1} ds + t C(M) \leq \sqrt{t} C(M).
\end{aligned}$$

Applying ∇ to (1.8) and (1.10), testing the results by $\nabla \sigma_t$ and $\nabla \theta_t$ in $L^2(\Omega \times (0,t))$, respectively, we derive

$$\begin{aligned}
&\int_0^t \int |\nabla \sigma_t|^2 dx ds + \frac{1}{\epsilon} \int_0^t \int \nabla \sigma_t \nabla \mathrm{div}\, u dx ds \\
&\quad = - \int_0^t \int \nabla \mathrm{div}\, (\sigma u) \nabla \sigma_t dx ds \leq t C(M), \quad (2.23)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2} \int |\Delta \theta|^2 dx + \int_0^t \int \rho |\nabla \theta_t|^2 dx ds + \frac{1}{\epsilon} \int_0^t \int \nabla \theta_t \nabla \mathrm{div}\, u dx ds \\
&= \frac{1}{2} \int |\Delta \theta_0|^2 dx + \int_0^t \int \epsilon \nabla [2\mu |D(u)|^2 + \lambda (\mathrm{div}\, u)^2 + (E + u \times b)^2] \nabla \theta_t dx ds \\
&\quad - \int_0^t \int \nabla [(\rho\theta + \sigma) \mathrm{div}\, u] \nabla \theta_t dx ds - \int_0^t \int [\nabla \rho \theta_t + \nabla(\rho u \cdot \nabla \theta)] \nabla \theta_t dx ds \\
&\leq C_0(M_0) + \sqrt{t} C(M). \quad (2.24)
\end{aligned}$$

Summing up (2.22), (2.23) and (2.24), we arrive at

$$\int(|\nabla \mathrm{div}\, u|^2+|\Delta\theta|^2)dx+\int_0^t\int(|\nabla\sigma_t|^2+|\nabla\theta_t|^2)dxds\le C_0(M_0)\exp(\sqrt{t}C(M)). \tag{2.25}$$

Testing (2.18), (2.19) and (2.20) by $-\Delta\sigma_t, -\nabla\mathrm{div}\, u_t$ and $-\Delta\theta_t$, respectively, we derive

$$\begin{aligned}&\frac{1}{2}\int|\nabla\sigma_t|^2dx+\frac{1}{2}\int_0^t\int\nabla\mathrm{div}\, u_t\nabla\sigma_t dxds\\ &\quad=\frac{1}{2}\int|\nabla\sigma_t(0)|^2dx+\int_0^t\int\mathrm{div}\,(\sigma_t u+\sigma u_t)\cdot\Delta\sigma_t dxds\\ &\quad=:\frac{1}{2}\int|\nabla\sigma_t(0)|^2dx+I_4.\end{aligned} \tag{2.26}$$

We bound I_4 as follows.

$$\begin{aligned}I_4=&\int_0^t\int u\nabla\sigma_t\Delta\sigma_t dx-\int_0^t\int\nabla(\sigma_t\mathrm{div}\, u+u_t\nabla\sigma+\sigma\mathrm{div}\, u_t)\nabla\sigma_t dxds\\ =&-\sum_i\int_0^t\int\partial_i u\nabla\sigma_t\partial_i\sigma_t dxds+\int_0^t\int\frac{1}{2}\mathrm{div}\, u|\nabla\sigma_t|^2dxds\\ &-\int_0^t\int\nabla(\sigma_t\mathrm{div}\, u+u_t\nabla\sigma+\sigma\mathrm{div}\, u_t)\nabla\sigma_t dxds\\ \le& C(M)\int_0^t\|\nabla u\|_{L^\infty}ds+C(M)\int_0^t\|u\|_{H^3}ds+C(M)\int_0^t\|u_t\|_{H^2}ds\\ \le&\sqrt{t}C(M).\end{aligned}$$

And

$$\begin{aligned}&\frac{1}{2}\int\rho(\mathrm{div}\, u_t)^2dx+(\lambda+2\mu)\int_0^t\int|\nabla\mathrm{div}\, u_t|^2dxds\\ &-\frac{1}{\epsilon}\int_0^t\int\nabla\mathrm{div}\, u_t(\nabla\sigma_t+\nabla\theta_t)dxds\\ =&\frac{1}{2}\int\rho_0(\mathrm{div}\, u_t(0))^2dx+\int_0^t\int[\rho_t u_t+(\rho u)_t\nabla u+\nabla(\sigma\theta)_t]\nabla\mathrm{div}\, u_t dxds\\ &+\int_0^t\int\mathrm{div}\,(E\times b)_t\mathrm{div}\, u_t dxds-\int_0^t\int[(u\times b)\times b]_t\nabla\mathrm{div}\, u_t dxds\\ &+\int_0^t\int(\epsilon\nabla\sigma\cdot u_{tt}+u\cdot\nabla u_t)\mathrm{div}\, u_t dxds-\int_0^t\int\sum_i\nabla u_i\partial_i u_t\mathrm{div}\, u_t dxds\\ \le& C_0(M_0)+\sqrt{t}C(M).\end{aligned} \tag{2.27}$$

And

$$\frac{1}{2}\int\rho|\nabla\theta_t|^2dx+\int_0^t\int|\Delta\theta_t|^2dxds+\frac{1}{\epsilon}\int_0^t\int\nabla\mathrm{div}\, u_t\nabla\theta_t dxds$$

$$
\begin{aligned}
=&\frac{1}{2}\int \rho_0|\nabla\theta_t(0)|^2dx - \int_0^t\int \epsilon\nabla\sigma\theta_{tt}\nabla\theta_t dxds - \int_0^t\int\sum_i \nabla(\rho u_i)\partial_i\theta_t\nabla\theta_t dxds\\
&+\int_0^t\int \Delta\theta[\rho_t\theta_t + (\rho u)_t\nabla\theta + ((\rho\theta+\sigma)\cdot\operatorname{div}u)_t]dxds\\
&-\epsilon\int_0^t\int \Delta\theta_t(2\mu|D(u)|^2 + \lambda(\operatorname{div}u)^2 + (E+u\times b)^2)_t dxds\\
\le& C_0(M_0) + \sqrt{t}C(M). \qquad (2.28)
\end{aligned}
$$

Summing up (2.26), (2.27) and (2.28), we arrive at

$$
\begin{aligned}
&\int(|\nabla\sigma_t|^2 + (\operatorname{div}u_t)^2 + |\nabla\theta_t|^2)dx + \int_0^t\int(|\nabla\operatorname{div}u_t|^2 + (\Delta\theta_t)^2)dxds\\
&\quad\le C_0(M_0)\exp(\sqrt{t}C(M)). \qquad (2.29)
\end{aligned}
$$

Now, testing $\partial_i\nabla$ (1.8) by $\partial_i\nabla\sigma + \partial_i\nabla\theta$ and the same calculations as those in [3] to obtain

$$
\begin{aligned}
&\frac{1}{2}\int|\partial_i\nabla\sigma|^2dx + \int\partial_i\nabla\sigma\cdot\partial_i\nabla\theta dx + \frac{1}{\epsilon}\int_0^t\int\partial_i\nabla\operatorname{div}u(\partial_i\nabla\sigma + \partial_i\nabla\theta)dxds\\
&\quad\le C_0(M_0) + \sqrt{t}C(M). \qquad (2.30)
\end{aligned}
$$

Testing ∂_i (1.9) by $\partial_i\nabla\operatorname{div}u$ in $L^2(\Omega\times(0,t))$ and the same calculations as those in [3] to obtain

$$
\frac{1}{2}\int_0^t\int|\partial_i\nabla\operatorname{div}u|^2dx - \frac{1}{\epsilon}\int_0^t\int\partial_i\nabla\operatorname{div}u(\partial_i\nabla\sigma + \partial_i\nabla\theta)dxds \le tC(M). \qquad (2.31)
$$

(2.30), (2.31) and (2.25) give

$$
\int|\nabla^2\sigma|^2dx + \int_0^t\int|\nabla^2\operatorname{div}u|^2dxds \le C_0(M_0)\exp(\sqrt{t}C(M)). \qquad (2.32)
$$

Applying rot to (1.9) and denoting the vorticity $\omega := \operatorname{rot}u$, we see that

$$
\begin{aligned}
\rho\omega_t + \rho u\cdot\nabla\omega - \mu\Delta\omega =&(\partial_j\rho u_{it} - \partial_i\rho u_{jt}) + (\partial_j(\rho u_k)\partial_k u_i - \partial_i(\rho u_k)\partial_k u_j)\\
&+\operatorname{rot}[(E+u\times b)\times b]. \qquad (2.33)
\end{aligned}
$$

We test (2.33) by $\Delta\omega$ in $L^2(\Omega\times(0,t))$ to get

$$
\int\rho|\operatorname{rot}\omega|^2dx + \mu\int_0^t\int|\Delta\omega|^2dxds \le C_0(M_0) + \sqrt{t}C(M). \qquad (2.34)
$$

Similarly, we apply ∂_t to (2.33) and test the resulting equations by ω_t in $L^2(\Omega\times(0,t))$ to deduce that

$$
\int\rho|\omega_t|^2dx + \mu\int_0^t\int|\operatorname{rot}\omega_t|^2dxds \le C_0(M_0) + \sqrt{t}C(M). \qquad (2.35)
$$

By the same calculations as that in [3], we have

$$\|\Delta\theta\|_{L^2(0,t;H^1)} \le C_0(M_0)\exp(t^{\frac{1}{4}}C(M)). \tag{2.36}$$

It follows from (1.11), (1.12), (2.6), (2.9), (2.10), (2.11), (2.13), (2.14), (2.15) and (2.16) that

$$\epsilon\|E_{tt}\|_{L^2(0,t;L^2)} \le \|\mathrm{rot}\, b_t - E_t - (u\times b)_t\|_{L^2(0,t;L^2)} \le C_0(M_0) + tC(M). \tag{2.37}$$

$$\sqrt{\epsilon}\|b_{tt}(t)\|_{L^2} \le \sqrt{\epsilon}\|\mathrm{rot}\, E_t(\cdot,t)\|_{L^2} \le C_0(M_0) + tC(M). \tag{2.38}$$

Finally, we need to estimate $\epsilon\sigma_{tt}$, ϵu_{tt} and $\epsilon\theta_{tt}$ to close the energy estimates. Testing $\partial_t^2(1.8)$, $\partial_t^2(1.9)$ and $\partial_t^2(1.10)$ by $\epsilon^2\sigma_{tt}$, $\epsilon^2 u_{tt}$ and $\epsilon^2\theta_{tt}$, respectively, then by the same calculations as that in [3], we conclude that

$$\epsilon\|(\sigma_{tt}, u_{tt}, \theta_{tt})(t)\|_{L^2} + \epsilon\|(u_{tt}, \theta_{tt})\|_{L^2(0,t;H^1)} \le C_0(M_0)\exp(t^{\frac{1}{4}}C(M))$$

and thus (1.19) hold true.

This completes the proof the proof of Theorem 1.1.

□

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